Exercise 1A

1 Let the velocity of P at time t be v m s^{-1}

$v = \int a\,dt = \int 3e^{-0.25t}\,dt = -12e^{-0.25t} + C$

When $t = 0$, $v = 4$

$4 = -12 + C \Rightarrow C = 16$

$v = 16 - 12e^{-0.25t}$

The velocity of the particle at time t seconds is $(16 - 12e^{-0.25t})$ m s^{-1}

2 Let the displacement of P from O at time t be x m.

$x = \int v\,dt = \int t\sin t\,dt$

Using integration by parts

$x = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C$

When $t = 0$, $x = 0$

$0 = 0 + 0 + C \Rightarrow C = 0$

$x = -t\cos t + \sin t$

When $t = \frac{\pi}{2}$

$x = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$

Hence P is 1 metre from O, as required.

3 Let the displacement of P from point A at time t be s m.

$s = \int v\,dt = \int \frac{4}{3+2t}\,dt = 2\ln(3+2t) + C$

When $t = 0$, $s = 0$

$0 = 2\ln 3 + C \Rightarrow C = -2\ln 3$

$s = 2\ln(3+2t) - 2\ln 3 = 2\ln\left(\frac{3+2t}{3}\right)$

When $t = 3$

$s = 2\ln\left(\frac{3+6}{3}\right) = 2\ln 3$

So $AB = 2\ln 3$ m

4 Let the velocity of P at time t be $v\,\text{m s}^{-1}$

$v = \int a\,dt = \int 4e^{\frac{1}{2}t}\,dt = 8e^{\frac{1}{2}t} + C$

When $t = 0, v = 0$

$0 = 8 + C \Rightarrow C = -8$

$v = 8e^{\frac{1}{2}t} - 8$

The distance moved in the interval $0 \leqslant t \leqslant 2$ is given by

$s = \int_0^2 v\,dt = \int_0^2 8e^{\frac{1}{2}t} - 8\,dt$

$= \left[16e^{\frac{1}{2}t} - 8t\right]_0^2 = (16e^1 - 16) - 16$

$= 16e - 32 = 11.5$ (3 s.f.)

The distance moved is 11.5 m (3 s.f.).

5 a Let the acceleration of P at time t be $a\,\text{m s}^{-2}$

$v = 4\cos 3t$

So $a = \frac{dv}{dt} = -12\sin 3t$

When $t = \frac{\pi}{12}$

$a = -12\sin\frac{\pi}{4} = -12 \times \frac{1}{\sqrt{2}} = -6\sqrt{2}$

The magnitude of the acceleration when $t = \frac{\pi}{12}$ is $6\sqrt{2}\,\text{m s}^{-2}$

b $x = \int v\,dt = \int 4\cos 3t\,dt = \frac{4}{3}\sin 3t + C$

When $t = 0, x = 0$

$0 = \frac{4}{3} \times 0 + C \Rightarrow C = 0$

So $x = \frac{4}{3}\sin 3t$

c When P is at O, $x = 0$

$x = \frac{4}{3}\sin 3t = 0 \Rightarrow \sin 3t = 0$

The smallest positive value of t is given by $3t = \pi \Rightarrow t = \frac{\pi}{3}$

6 $v = \int a \, dt = \int \frac{6t}{(2+t^2)^2} dt$

Using integration by substitution, let $u = 2 + t^2$, so $\frac{du}{dt} = 2t$

$v = \int \frac{6t}{(2+t^2)^2} dt = \int \frac{3}{(2+t^2)^2} \times 2t \, dt$

$= \int \frac{3}{u^2} du = \int 3u^{-2} du$

$= -3u^{-1} + C = C - \frac{3}{u}$

$= C - \frac{3}{2+t^2}$

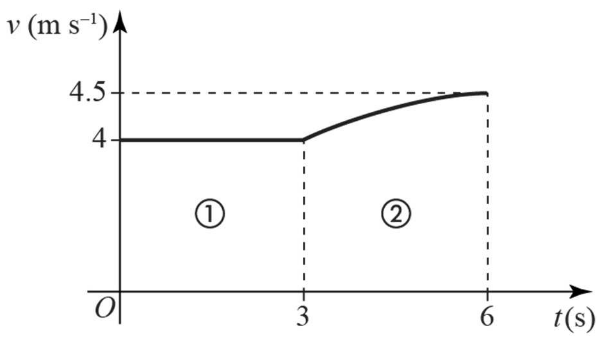
When $t = 0$, $v = 0$

$0 = C - \frac{3}{2} \Rightarrow C = \frac{3}{2}$

So $v = \frac{3}{2} - \frac{3}{2+t^2}$

7 a For $v = 4$, the graph is a straight line from (0, 4) to (3, 4).

For $v = 5 - \frac{3}{t}$, the graph is part of a reciprocal curve joining (3, 4) to (6, 0.5)



b The distance moved in the first three seconds is represented by the area labelled (1).
Let this area be A_1. Then $A_1 = 3 \times 4 = 12$

The distance travelled in the next three seconds is represented by the area labelled (2).
Let this area be A_2.

$A_2 = \int_3^6 \left(5 - \frac{3}{t}\right) dt = \left[5t - 3\ln t\right]_3^6 = (30 - 3\ln 6) - (15 - 3\ln 3)$

$= 15 - 3(\ln 6 - \ln 3) = 15 - 3\ln 2$

So the displacement of P from O when $t = 6$ is $(12 + 15 - 3\ln 2)\,\text{m} = (27 - 3\ln 2)\,\text{m}$.

8 a $v = \int a\,\mathrm{d}t = \int \sin\frac{1}{2}t\,\mathrm{d}t = -2\cos\frac{1}{2}t + C$

When $t = 0,\ v = 0$

$0 = -2 + C \Rightarrow C = 2$

$v = 2 - 2\cos\frac{1}{2}t$

When $t = 2\pi$

$v = 2 - 2\cos\pi = 2 - (2\times -1) = 4$

The speed of P when $t = 2\pi$ is $4\ \mathrm{m\,s^{-1}}$

b $x = \int v\,\mathrm{d}t = \int\left(2 - 2\cos\frac{1}{2}t\right)\mathrm{d}t = 2t - 4\sin\frac{1}{2}t + B$

When $t = 0,\ x = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

$x = 2t - 4\sin\frac{1}{2}t$

When $t = \frac{\pi}{2}$

$x = 2\times\frac{\pi}{2} - 4\sin\frac{\pi}{4} = \pi - 4\times\frac{1}{\sqrt{2}} = \pi - 2\sqrt{2}$

The displacement of P from O when $t = \frac{\pi}{2}$ is $(\pi - 2\sqrt{2})$ m.

9 a $v = \int a\,\mathrm{d}t = \int -4\mathrm{e}^{0.2t}\,\mathrm{d}t = -20\mathrm{e}^{0.2t} + C$

When $t = 0,\ v = 20$

$20 = -20 + C \Rightarrow C = 40$

$v = 40 - 20\mathrm{e}^{0.2t}$

b $x = \int v\,\mathrm{d}t = \int (40 - 20\mathrm{e}^{0.2t})\,\mathrm{d}t = 40t - 100\mathrm{e}^{0.2t} + B$

When $t = 0,\ x = 0$

$0 = 0 - 100 + B \Rightarrow B = 100$

$x = 40t - 100\mathrm{e}^{0.2t} + 100$

The maximum value of x occurs when $\frac{\mathrm{d}x}{\mathrm{d}t} = v = 40 - 20\mathrm{e}^{0.2t} = 0$

$\Rightarrow \mathrm{e}^{0.2t} = 2$

$\Rightarrow 0.2t = \ln 2$

$\Rightarrow t = 5\ln 2$

So the maximum value of x

$= 40\times 5\ln 2 - 100\times\mathrm{e}^{0.2\times 5\ln 2} + 100 = 200\ln 2 - 100\mathrm{e}^{\ln 2} + 100$

$= 200\ln 2 - 200 + 100 = 200\ln 2 - 100$

The maximum displacement of P from O in the direction of x-increasing is $(200\ln 2 - 100)$ m.

10 a $v = \dfrac{3200}{c + kt}$

When $t = 0$, $v = 40$

$$40 = \frac{3200}{c} \Rightarrow c = 80$$

So $v = \dfrac{3200}{80 + kt} = 3200(80 + kt)^{-1}$

Hence $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -3200k(80 + kt)^{-2} = -\dfrac{3200k}{(80 + kt)^2}$

When $t = 0$, $a = -0.5$

$$-0.5 = -\frac{3200k}{80^2} \Rightarrow k = \frac{0.5 \times 80^2}{3200} = 1$$

Solution: $c = 80$, $k = 1$

b $x = \int v \, \mathrm{d}t = \int \dfrac{3200}{80 + t} \mathrm{d}t = 3200 \ln(80 + t) + A$

When $t = 0$, $x = 0$

$$0 = 3200 \ln 80 + A \Rightarrow A = -3200 \ln 80$$

$$x = 3200 \ln(80 + t) - 3200 \ln 80 = 3200 \ln\left(\frac{80 + t}{80}\right)$$

11 a $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 2\mathrm{e}^{2t} - 11\mathrm{e}^{t} + 15$

When $a = 0$

$$2\mathrm{e}^{2t} - 11\mathrm{e}^{t} + 15 = 0$$

$$(2\mathrm{e}^{t} - 5)(\mathrm{e}^{t} - 3) = 0$$

$$\mathrm{e}^{t} = 2.5, 3$$

$$t = \ln 2.5, \ln 3$$

b $x = \int v \, \mathrm{d}t = \int (\mathrm{e}^{2t} - 11\mathrm{e}^{t} + 15t) \mathrm{d}t = \dfrac{\mathrm{e}^{2t}}{2} - 11\mathrm{e}^{t} + \dfrac{15t^2}{2} + C$

When $t = 0$, $x = 0$

$$0 = \frac{1}{2} - 11 + 0 + C \Rightarrow C = \frac{21}{2}$$

$$x = \frac{\mathrm{e}^{2t}}{2} - 11\mathrm{e}^{t} + \frac{15t^2}{2} + \frac{21}{2}$$

When $t = \ln 3$

$$x = \frac{\mathrm{e}^{2\ln 3}}{2} - 11\mathrm{e}^{\ln 3} + \frac{15(\ln 3)^2}{2} + \frac{21}{2}$$

$$= \frac{9}{2} - 33 + \frac{15(\ln 3)^2}{2} + \frac{21}{2} = \frac{15(\ln 3)^2}{2} - 18 \approx -8.95$$

As displacement is a positive quantity, the required distance is $\left(18 - \dfrac{15(\ln 3)^2}{2}\right)$ m.

12 a $a = \frac{dv}{dt} = 2 + \frac{1}{t+2}$

When $a = 2.2$

$$2 + \frac{1}{t+2} = 2.2 \Rightarrow t + 2 = \frac{1}{0.2} = 5 \Rightarrow t = 3$$

Note that if $a = -2.2$, which also has a magnitude of $2.2\ \text{m s}^{-2}$, then this gives

$$2 + \frac{1}{t+2} = -2.2 \Rightarrow t + 2 = -\frac{1}{4.2} \Rightarrow t \approx -2.24$$

So this is not a valid result as $t > 0$.

b $x = \int v\,dt = \int_1^4 (2t + \ln(t+2))\,dt$

Using integration by parts to work out the second term of the integral, with $u = \ln(t+2)$ and $v = t$

$$\int \ln(t+2)\,dt = \int 1 \times \ln(t+2)\,dt$$

$$= t\ln(t+2) - \int \frac{t}{t+2}\,dt = t\ln(t+2) - \int\left(1 - \frac{2}{t+2}\right)dt$$

$$= t\ln(t+2) - t + 2\ln(t+2) = (t+2)\ln(t+2) - t$$

Hence $x = \left[t^2 + (t+2)\ln(t+2) - t\right]_1^4$

$$= (16 + 6\ln 6 - 4) - (1 + 3\ln 3 - 1)$$

$$= 12 + 6\ln 6 - 3\ln 3 = 12 + 3\ln 6^2 - 3\ln 3$$

$$= 12 + 3\ln\left(\frac{36}{3}\right) = 12 + 3\ln 12$$

So the distance moved by P in the interval $1 \leqslant t \leqslant 4$ is $(12 + 3\ln 12)$ m.

13 a $v = 3t^2 - 5t + 2$

When $v = 0$

$$3t^2 - 5t + 2 = 0$$

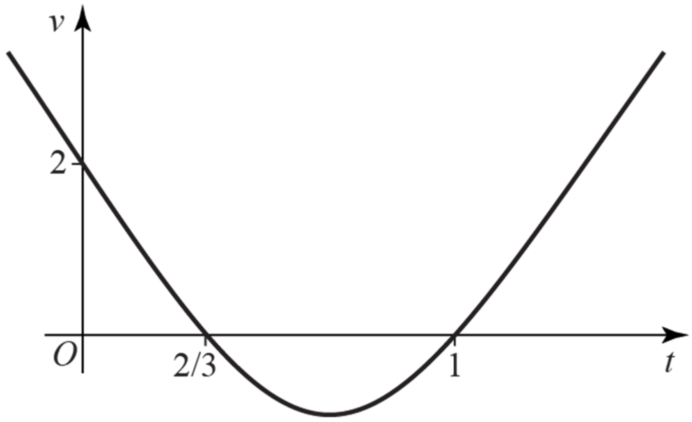
$$(3t - 2)(t - 1) = 0$$

$$t = \frac{3}{2}, t = 1$$

b $a = \frac{dv}{dt} = 6t - 5$

When $t = 5$, $a = 6 \times 5 - 5 = 25\ \text{m s}^{-2}$

13 c The velocity-time graph for the motion of the particle is:



The particle changes direction twice in the interval $0 \leqslant t \leqslant 5$

$$\text{Total distance travelled} = \int_0^{\frac{2}{3}} (3t^2 - 5t + 2)\,\mathrm{d}t - \int_{\frac{2}{3}}^{1} (3t^2 - 5t + 2)\,\mathrm{d}t + \int_1^5 (3t^2 - 5t + 2)\,\mathrm{d}t$$

$$= \left[t^3 - \frac{5}{2}t^2 + 2t\right]_0^{\frac{2}{3}} - \left[t^3 - \frac{5}{2}t^2 + 2t\right]_{\frac{2}{3}}^{1} + \left[t^3 - \frac{5}{2}t^2 + 2t\right]_1^5$$

$$= \left(\frac{8}{27} - \frac{10}{9} + \frac{4}{3}\right) - \left(\frac{1}{2} - \frac{8}{27} + \frac{10}{9} - \frac{4}{3}\right) + \left(125 - \frac{125}{2} + 10 - \frac{1}{2}\right)$$

$$= \frac{14}{27} + \frac{1}{54} + \frac{144}{2} = \frac{28 + 1 + 3888}{54} = \frac{3917}{54} = 72.5 \text{ m (3 s.f.)}$$

d Let the displacement of P from O at time t be x m.

$$x = \int v\,dt = \int (3t^2 - 5t + 2)\,\mathrm{d}t = t^3 - \frac{5}{2}t^2 + 2t + C$$

When $t = 0$, $x = 0 \Rightarrow C = 0$

Therefore $x = t^3 - \frac{5}{2}t^2 + 2t = t\left(t^2 - \frac{5}{2}t + 2\right)$

If P returns to O, then $x = 0$ for some value of t ($t > 0$).

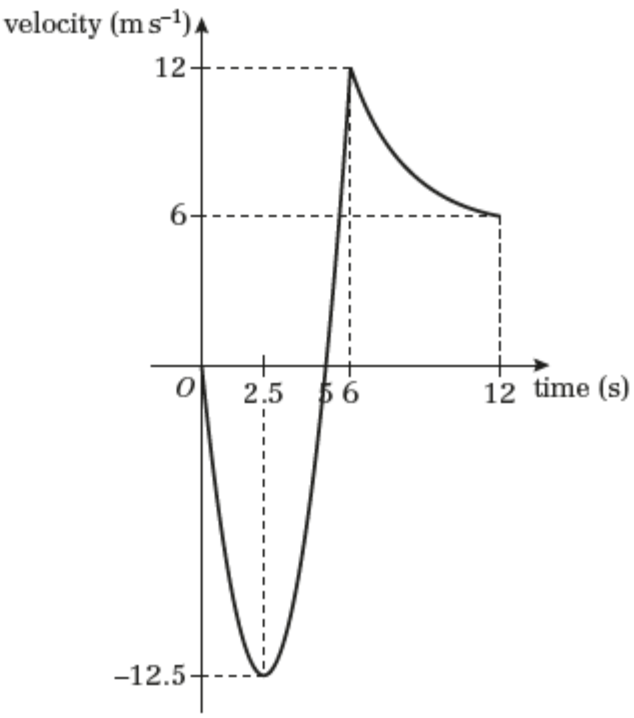
Looking for solutions of $t^2 - \frac{5}{2}t + 2 = 0$

The discriminant $(b^2 - 4ac)$ of this expression is $\left(\frac{5}{2}\right)^2 - 8 = \frac{25}{4} - \frac{32}{4} = -\frac{7}{4}$

As the discriminant is negative, this expression has no real roots. Therefore P never returns to O for any $t > 0$.

14 a For $v = 2t(t-5)$, the graph is a quadratic, with a positive x^2 coefficient, from (0, 0) to (6, 12). It cuts the x-axis at (0, 0) and (5, 0) and has a minimum at (2.5, –12.5)

For $v = \dfrac{72}{t}$, the graph is part of a reciprocal curve joining (6, 12) to (12, 6)



b For $0 \leqslant t \leqslant 6,\ a = \dfrac{dv}{dt} = 4t - 10$ This is positive when $4t < 10$, i.e. $2.5 < t \leqslant 6$

For $6 < t \leqslant 12,\ a = \dfrac{dv}{dt} = -\dfrac{72}{t^2}$ This is negative for all values of t

So the acceleration is positive for $2.5 < t \leqslant 6$

c The definite integral will be negative for the area below the x-axis in the graph in part **a**

$$\text{Total distance travelled} = -\int_0^5 (2t^2 - 10t)\,dt + \int_5^6 (2t^2 - 10t)\,dt + \int_6^{12} \frac{72}{t}\,dt$$

$$= -\left[\frac{2}{3}t^3 - 5t^2\right]_0^5 + \left[\frac{2}{3}t^3 - 5t^2\right]_5^6 + \left[72\ln t\right]_6^{12}$$

$$= \frac{125}{3} - 36 + \frac{125}{3} + 72\ln 12 - 72\ln 6$$

$$= \frac{250}{3} - \frac{108}{3} + 72(\ln 12 - \ln 6) = \left(\frac{142}{3} + 72\ln 2\right)\text{ m}$$

Challenge

$a = \frac{dv}{dt} = \frac{60}{kt^2}$ for $t \geqslant 2$

So $v = \int \frac{60}{kt^2}\, dt = -\frac{60}{kt} + C$

When $t = 2,\ v = 0$

$0 = -\frac{60}{2k} + C \Rightarrow C = \frac{30}{k}$

When $t = 5,\ v = 9$

$9 = -\frac{60}{5k} + C \Rightarrow C = 9 + \frac{12}{k}$

So $9 + \frac{12}{k} = \frac{30}{k} \Rightarrow 9 = \frac{18}{k} \Rightarrow k = 2$

And $C = \frac{30}{k} \Rightarrow C = 15$

So $v = 15 - \frac{30}{t}$ for $t \geqslant 2$

As $0 < \frac{30}{t} \leqslant 15$ for $t \geqslant 2$, $|v| < 15$

So for $t \geqslant 2$ the car never reaches a speed of 15 ms^{-1}

Exercise 1B

1 $a = 2 + \frac{1}{2}x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 + \frac{1}{2}x$

$\frac{1}{2}v^2 = \int\left(2 + \frac{1}{2}x\right)dx = 2x + \frac{x^2}{4} + A$

At $x = 0, v = 5$

$\frac{1}{2} \times 25 = 0 + 0 + A \Rightarrow A = \frac{25}{2}$

$\frac{1}{2}v^2 = 2x + \frac{x^2}{4} + \frac{25}{2}$

$v^2 = \frac{x^2}{2} + 4x + 25$

2 $a = -4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$

$\frac{1}{2}v^2 = \int(-4x)\,dx = -2x^2 + A$

At $x = 2, v = 8$

$\frac{1}{2} \times 64 = -8 + A \Rightarrow A = 40$

$\frac{1}{2}v^2 = -2x^2 + 40$

$v^2 = 80 - 4x^2$

$v = \pm\sqrt{(80 - 4x^2)}$

3 $a = \frac{4}{x^2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4}{x^2}$

$\frac{1}{2}v^2 = \int(4x^{-2})\,dx = -4x^{-1} + A = A - \frac{4}{x}$

At $x = 2, v = 6$

$\frac{1}{2} \times 36 = A - 2 \Rightarrow A = 20$

$\frac{1}{2}v^2 = 20 - \frac{4}{x}$

When $v = 0$

$0 = 20 - \frac{4}{x} \Rightarrow x = \frac{4}{20} = \frac{1}{5}$

4 $a = -25x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -25x$

$\frac{1}{2}v^2 = \int(-25x)\,dx = -\frac{25}{2}x^2 + A$

At $x = 0$, $v = 40$

$\frac{1}{2} \times 1600 = -0 + A \Rightarrow A = 800$

$\frac{1}{2}v^2 = -\frac{25}{2}x^2 + 800$

$v^2 = 1600 - 25x^2$

When $v = 0$

$25x^2 = 1600 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$

So $AB = 16$ m

5 a $a = -kx^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -kx^2$

$\frac{1}{2}v^2 = \int(-kx^2)\,dx = -\frac{kx^3}{3} + A$

At $x = 0$, $v = 16$

$\frac{1}{2} \times 256 = -0 + A \Rightarrow A = 128$

$\frac{1}{2}v^2 = -\frac{kx^3}{3} + 128$

When $v = 0$, $x = 20$

$0 = -\frac{8000k}{3} + 128 \Rightarrow k = \frac{3 \times 128}{8000} = \frac{3 \times 16}{1000} = \frac{6}{125}$

b From part **a**, $\frac{1}{2}v^2 = -\frac{6}{125} \times \frac{x^2}{3} + 128 \Rightarrow v^2 = 256 - \frac{4}{125}x^3$

At $x = 10$

$v^2 = 256 - \frac{4}{125} \times 1000 = 256 - 32 = 224$

$v = \pm\sqrt{224} = \pm 4\sqrt{14}$

The velocity of P at $x = 10$ is $\pm 4\sqrt{14}$ m s^{-1} as the particle will pass through this position in both directions.

6 $a = -8x^3$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -8x^3$$

$$\frac{1}{2}v^2 = \int(-8x^3)\,\mathrm{d}x = -2x^4 + A$$

At $x = 2$, $v = 32$

$$\frac{1}{2}\times 1024 = A - 32 \Rightarrow A = 544$$

$$\frac{1}{2}v^2 = 544 - 2x^4$$

$$v^2 = 1088 - 4x^4$$

When $v = 8$

$$64 = 1088 - 4x^4 \Rightarrow x^4 = 256$$

$$x = 256^{\frac{1}{4}} = 4$$

7 a $a = 6\sin\frac{x}{3}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 6\sin\frac{x}{3}$$

$$\frac{1}{2}v^2 = \int\left(6\sin\frac{x}{3}\right)\mathrm{d}x = -18\cos\frac{x}{3} + A$$

At $x = 0$, $v = 4$

$$\frac{1}{2}\times 16 = -18 + A \Rightarrow A = 26$$

$$\frac{1}{2}v^2 = -18\cos\frac{x}{3} + 26$$

$$v^2 = 52 - 36\cos\frac{x}{3}$$

b The greatest value of v^2 occurs when $\cos\frac{x}{3} = -1$

The greatest value of v^2 is given by $v^2 = 52 + 36 = 88 \Rightarrow v = \pm\sqrt{88} = \pm 2\sqrt{22}$

So the greatest possible speed of P is $2\sqrt{22}\ \mathrm{m\,s^{-1}}\ (\approx 9.38\ \mathrm{m\,s^{-1}})$

8 $a = 2 + 3e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 + 3e^{-x}$

$\frac{1}{2}v^2 = \int (2 + 3e^{-x})\,dx = 2x - 3e^{-x} + A$

At $x = 0$, $v = 2$

$\frac{1}{2} \times 4 = 0 - 3 + A \Rightarrow A = 5$

$\frac{1}{2}v^2 = 2x - 3e^{-x} + 5$

$v^2 = 4x - 6e^{-x} + 10$

At $x = 3$

$v^2 = 12 - 6e^{-3} + 10 = 21.701$ (3 d.p.)

$v = \sqrt{21.701\ldots} = 4.658$ (3 d.p.)

The velocity of P at $x = 3$ is 4.66 m s^{-1} (3 s.f.), in the direction of x increasing.

9 a $a = -\frac{4}{2x+1}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{4}{2x+1}$

$\frac{1}{2}v^2 = \int\left(-\frac{4}{2x+1}\right)dx = -2\ln(2x+1) + A$

At $x = 0$, $v = 4$

$\frac{1}{2} \times 16 = -0 + A \Rightarrow A = 8$

$\frac{1}{2}v^2 = -2\ln(2x+1) + 8$

$v^2 = 16 - 4\ln(2x+1)$

At $x = 10$

$v^2 = 16 - 4\ln 21 = 3.8219$ (4 d.p.)

$v = 1.95\,\text{m s}^{-1}$ (3 s.f.)

The speed of P at $x = 10$ is $1.95\,\text{m s}^{-1}$ (3 s.f.)

b When $v = 0$

$0 = 16 - 4\ln(2x+1) \Rightarrow \ln(2x+1) = 4$

So $2x + 1 = e^4 \Rightarrow x = \frac{e^4 - 1}{2} = 26.8$ (3 s.f.)

10 a $a = x - \dfrac{4}{x^3}$

$\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = x - \dfrac{4}{x^3}$

$\dfrac{1}{2}v^2 = \int (x - 4x^{-3})\,dx = \dfrac{x^2}{2} + 2x^{-2} + A = \dfrac{x^2}{2} + \dfrac{2}{x^2} + A$

At $x = 1$, $v = 3$

$\dfrac{1}{2} \times 9 = \dfrac{1}{2} + 2 + A \Rightarrow A = 2$

$\dfrac{1}{2}v^2 = \dfrac{x^2}{2} + \dfrac{2}{x^2} + 2$

$v^2 = x^2 + 4 + \dfrac{4}{x^2} = \left(x + \dfrac{2}{x}\right)^2$

$v = x + \dfrac{2}{x}$

b The minimum value of v occurs when $\dfrac{dv}{dt} = a = 0$

$x - \dfrac{4}{x^3} = 0 \Rightarrow x^4 = 4 \Rightarrow x = \sqrt{2}$ (as P moves on the positive x-axis, $x > 0$)

At $x = \sqrt{2}$

$v = \sqrt{2} + \dfrac{2}{\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

The least speed of P during its motion is $2\sqrt{2}\,\text{m s}^{-1}$

11 $a = -\left(10 + \dfrac{1}{4}x\right)$

$\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -10 - \dfrac{1}{4}x$

$\dfrac{1}{2}v^2 = \int\left(-10 - \dfrac{1}{4}x\right)dx = -10x - \dfrac{x^2}{8} + A$

At $x = 0$, $v = 15$

$\dfrac{1}{2} \times 225 = -0 - 0 + A \Rightarrow A = \dfrac{225}{2}$

$\dfrac{1}{2}v^2 = -10x - \dfrac{x^2}{8} + \dfrac{225}{2}$

$v^2 = 225 - 20x - \dfrac{x^2}{4} = -\dfrac{x^2 + 80x - 900}{4} = -\dfrac{(x+90)(x-10)}{4}$

$v = 0 \Rightarrow x = 10, -90$

As P is initially moving in the direction of x increasing, it reaches $x = 10$ before $x = -90$. The distance P moves before first coming to instantaneous rest is 10 m.

12 a $a = 6x^{\frac{1}{3}}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^{\frac{1}{3}}$$

$$\frac{1}{2}v^2 = \int 6x^{\frac{1}{3}}\,dx = \frac{6x^{\frac{4}{3}}}{\frac{4}{3}} + A = \frac{9}{2}x^{\frac{4}{3}} + A$$

$v^2 = 9x^{\frac{4}{3}} + B$, where $B = 2A$

At $x = 8$, $v = 12$

$$144 = 9\times 16 + B \Rightarrow B = 0$$

$$v^2 = 9x^{\frac{4}{3}}$$

$$v = 3x^{\frac{2}{3}}$$

b $v = \frac{dx}{dt} = 3x^{\frac{2}{3}}$

Separating the variables and integrating

$$\int x^{-\frac{2}{3}}\,dx = \int 3\,dt$$

$$3x^{\frac{1}{3}} = 3t + C$$

When $t = 0$, $x = 8$

$$3\times 2 = 0 + C \Rightarrow C = 6$$

$$3x^{\frac{1}{3}} = 3t + 6$$

$$x^{\frac{1}{3}} = t + 2$$

$$x = (t+2)^3$$

Challenge

$$a = \frac{1}{10}(25 - x)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{25}{10} - \frac{x}{10}$$

$$\frac{1}{2}v^2 = \int\left(\frac{25}{10} - \frac{x}{10}\right)dx = \frac{25x}{10} - \frac{x^2}{20} + A$$

The maximum value of v occurs when $\frac{dv}{dt} = a = 0$, $a = \frac{1}{10}(25 - x) = 0 \Rightarrow x = 25$

So at $x = 25$, $v = 12$

$$\frac{1}{2}12^2 = \frac{25\times 25}{10} - \frac{25^2}{20} + A \Rightarrow A = 72 - \frac{625}{20} = \frac{288-125}{4} = \frac{163}{4}$$

Hence $\frac{1}{2}v^2 = \frac{25x}{10} - \frac{x^2}{20} + \frac{163}{4}$, so $v^2 = \frac{25x}{5} - \frac{x^2}{10} + \frac{163}{2} = \frac{1}{5}\left(25x - \frac{x^2}{2}\right) + \frac{163}{2}$